Spontaneous Magnetic Flux and Quantum Noise in a Doubly Connected Mesoscopic SND Junction

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We consider spontaneous magnetic flux in a doubly connected s-wave/normal metal/d-wave superconductor (SND) Josephson junction. The flux magnitude is not quantized and is determined by the parameters of the system. In an isolated system, quantum fluctuations of the superconducting phase take place, leading to quantum flux noise. The possibilities of experimental observations of the effect are discussed.

Mesoscopic SNS systems containing superconductors with different pairing symmetry promise a whole new range of phenomena, such as half-periodic Josephson effect [1] and spontaneous currents parallel to the interface [2]. In the latter paper, it was shown that due to local violation of $\mathcal{T}$-symmetry, spontaneous currents should flow parallel to the SN interface in the normal part of the system; the authors noted that such a current is difficult to observe directly, since it will average to zero over short distances.

This difficulty does not appear in a doubly connected SND system. There the spontaneous current would give rise to an equilibrium magnetic flux. Its magnitude is less than Φ_0 and is determined only by the parameters of the system. It should not be mixed up with the fractional flux that is predicted in a contour with a $\mathcal{T}$-symmetry-violating junction [2–4]. The latter is created by the current flow *across* the junction; the former is directly created by the spontaneous current flowing *parallel* to the SN interface. It is therefore an analog of persistent currents in normal mesoscopic rings [5], when the $\mathcal{T}$-symmetry is broken by an external magnetic flux.

We consider here an annular SND junction (Fig.1). This idealised structure consists of the inner (s-wave) superconducting contact S, conducting ring N, and the outer (d-wave) superconducting ring D. The splits in S and D ensure that the screening supercurrents in these electrodes do not mask the effects of spontaneous current in the normal part of the system, N. For the sake of simplicity, we assume that the orientation of the d-wave order parameter positive and negative lobes with respect to the radius vector is the same along the d-wave ring.

We calculate Josephson current in the system, following [1]. It is carried through the normal region by two sets of Andreev levels, coupled to positive and negative lobes of the d-wave order parameter (chosen to be real; the phase difference across the contact is thus ϕ, where $\Delta_1 e^{i\phi}$ is the order parameter of the s-wave superconductor). We are considering a "small" junction limit, where the influence on the Josephson current of its own magnetic field can be neglected. This imposes a condition (see Fig.1)

$$2\pi R < \lambda_J, \tag{1}$$

where $\lambda_J = \sqrt{\hbar c^2/8\pi e j_c \Lambda}$ is the Josephson penetration length [6]; j_c is the critical Josephson current density, and $\Lambda = W + \lambda_{L,1} + \lambda_{L,2}$ is the integral London penetration length, $\lambda_{L,a}$ being the penetration length in the ath superconductor. In case of SNS junction

$$\lambda_J \approx \lambda_F \sqrt{\frac{W}{\Lambda}\frac{c/v_F}{4\pi\alpha_{fs}}} < \lambda_F \sqrt{\frac{c/v_F}{4\pi\alpha_{fs}}} \equiv \lambda_F \zeta, \tag{2}$$

where λ_F is Fermi wavelength in the normal part of the system, and $\alpha_{fs} \equiv e^2/\hbar c \approx 1/137$.

Using the set of cylindric coordinates (r, α, z), we write the Bogoliubov-de Gennes equations in the normal region for a mode with vertical momentum k_z as

$$\begin{pmatrix} -\frac{1}{2m^*}\nabla^2 - \mu_\perp & 0 \\ 0 & \frac{1}{2m^*}\nabla^2 + \mu_\perp \end{pmatrix} \begin{pmatrix} u(r,\alpha;k_z) \\ v(r,\alpha;k_z) \end{pmatrix} = E \begin{pmatrix} u(r,\alpha;k_z) \\ v(r,\alpha;k_z) \end{pmatrix}, \tag{3}$$

where $\mu_\perp = (k_F^2 - k_z^2)/2m^* \equiv k_\perp^2/2m^*$, and m^* is the effective mass of the electron. Expanding u, v over asymuthal modes, $u(r,\alpha) = \sum_n U_n^+(r) e^{in\alpha}/\sqrt{r}$, $v(r,\alpha) = \sum_n U_n^-(r) e^{in\alpha}/\sqrt{r}$, we find:

$$-\frac{d^2}{dr^2}U_n^\pm(r) + \frac{n^2 - 1/4}{r^2}U_n^\pm(r) = (\pm 2m^* E + k_\perp^2) U_n^\pm(r). \tag{4}$$

The Andreev levels are determined from the quantization condition [7]

$$\oint p\,dr = \int_R^{R+W} dr\, p(r,E) + \int_{R+W}^{R} dr\, p(r,-E) = 2\pi m + \pi \pm (\phi + \delta_{n,k_z}), \tag{5}$$

where m is integer, and $p(r,E) = (k_\perp^2 + (n^2 - 1/4)/r^2 + 2m^* E)^{1/2}$. The intrinsic phase shift δ_{n,k_z} in mode (n,k_z) is zero if the corresponding lobe of d-wave order parameter carries positive sign, and π otherwise [1]. The radial and asymuthal currents in a n, k_z-mode are now easily found (cf. [1,2]) as:

$$J^{r(a)}_{n,k_z}(\phi + \delta_{n,k_z}) = j^{r(a)}_{n,k_z} F(\frac{\pi}{\beta\epsilon(n,k_z)}, \frac{1}{\tau\epsilon(n,k_z)}; \phi + \delta_{n,k_z}). \tag{6}$$

Here $j^{r,(a)}_{n,k_z}$ is the partial radial (asymuthal) current carried by Andreev levels with energies $E \approx 0$:

$$j^r_{n,k_z} = e\epsilon(n,k_z);\; j^a_{n,k_z} = \frac{e\epsilon(n,k_z)n}{2\pi(n^2-1/4)1/2}\left(\arctan\left[\frac{(k_\perp(R+W))^2}{n^2-1/4} - 1\right] - \arctan\left[\frac{(k_\perp R)^2}{n^2-1/4} - 1\right]\right); \tag{7}$$

and

$$\epsilon(n,k_z) = \frac{k_\perp^2/2m^*}{[k_\perp^2(R+W)^2 - (n^2-1/4)]^{1/2} - [k_\perp^2 R^2 - (n^2-1/4)]^{1/2}} \tag{8}$$

plays the role of the interlevel spacing $v_F^{\parallel}/2W$ of a planar contact. The currents depend on the phase through

$$F(\frac{\pi}{\beta\epsilon(n,k_z)}, \frac{1}{\tau\epsilon(n,k_z)}; \phi + \delta_{n,k_z}) = \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\sin m(\phi+\delta_{n,k_z})}{\sinh\frac{\pi m}{\beta\epsilon(n,k_z)}} \frac{\pi e^{-m/\tau\epsilon(n,k_z)}}{\beta\epsilon(n,k_z)}, \tag{9}$$

where $\tau \gg \epsilon(n,k_z)^{-1}$ is the elastic scattering time. In the limit of zero temperature and no scattering $F(0,0;\phi)$ becomes the 2π-periodic sawtooth of unit amplitude.

First we find the Josephson (radial) current:

$$I_J(\phi) = \sum_{n,k_z}^{+} J_r(n,k_z;\phi) + \sum_{n,k_z}^{-} J_r(n,k_z;\phi+\pi). \tag{10}$$

The "$\pm$"-sums are taken over zero- ($\delta_{n,k_z} = 0$), and π-levels ($\delta_{n,k_z} = \pi$), coupled to positive and negative lobes of the d-wave order parameter respectively [1]. At zero temperature and without elastic scattering this reduces to

$$I_J(\phi) = \frac{1+Z}{2} I_0 F(0,0;\phi) + \frac{1-Z}{2} I_0 F(0,0;\phi+\pi), \tag{11}$$

where $I_0 = N_\perp e\bar{\epsilon}$ is the critical current in the system, $N_\perp \sim \mathcal{A}/\lambda_F^2$ is the number of quantum conducting channels through the normal part of the system (of crosssection area $\mathcal{A}$), and $\bar{\epsilon} \equiv \langle\epsilon(n,k_z)\rangle \sim v_F/2W$ is the average interlevel spacing. The imbalance coefficient Z ($|Z| \le 1$) depends on the orientation of d-wave superconductor:

$$Z = \left\{\sum_{n,k_z}^{+}\epsilon(n,k_z) - \sum_{n,k_z}^{-}\epsilon(n,k_z)\right\} \Bigg/ \left\{\sum_{n,k_z}^{+}\epsilon(n,k_z) + \sum_{n,k_z}^{-}\epsilon(n,k_z)\right\}, \tag{12}$$

and determines the equilibrium phase difference across the junction, $\phi_0 = \pm\frac{1-Z}{2}\pi$. The Josephson energy of the system is thus

$$U(\phi) = \frac{1}{2e}\int d\phi I_J(\phi) = \frac{I_0}{2e}\frac{(|\phi| - \phi_0)^2}{2\pi^2} \quad (-\pi \le \phi \le \pi). \tag{13}$$

In equilibrium, the total Josephson current is zero, because contributions from zero- and π-levels cancel. On the contrary, their contributions to the asymuthal current add up and lead to finite spontaneous current. In the situation of Fig.1, due to partial cancellation of terms with positive and negative n, it equals

$$I_s(\phi) = \begin{cases} \sum_{k_z}\sum_{n=n^-(\theta,k_z)+1}^{n^+(\theta,k_z)} (J_a(|n|,k_z;\phi) - J_a(|n|,k_z;\phi+\pi)) & \text{if } n^+(\theta,k_z) > n^-(\theta,k_z); \\ 0 & \text{if } n^+(\theta,k_z) = n^-(\theta,k_z); \\ -\sum_{k_z}\sum_{n=n^+(\theta,k_z)+1}^{n^-(\theta,k_z)} (J_a(|n|,k_z;\phi) - J_a(|n|,k_z;\phi+\pi)) & \text{if } n^+(\theta,k_z) < n^-(\theta,k_z). \end{cases} \tag{14}$$

Here ($[x]$ means integral part of x)

$$n^+(\theta, k_z) = \left[\frac{\tan\theta\sqrt{(k_\perp R_2)^2 + 1/4}}{\sqrt{1 + \tan^2\theta(k_\perp R_2)^2}}\right];\; n^-(\theta, k_z) = \left[\frac{\cot\theta\sqrt{(k_\perp R_2)^2 + 1/4}}{\sqrt{1 + \cot^2\theta(k_\perp R_2)^2}}\right]. \quad (15)$$

In the limit $\beta, \tau, R, R + W \to \infty$ we return to the result [2] for a planar junction [8],

$$I_s(\phi) = \frac{\pi e v_F}{\sqrt{2}W}\frac{\mathcal{A}}{\lambda_F^2}\sin(\theta - \frac{\pi}{4})(F(0, 0; \phi) - F(0, 0; \phi + \pi)); \quad 0 \leq \theta \leq \pi/2. \quad (16)$$

In the annular junction spontaneous current produces a spontaneous magnetic flux, $\Phi_s = \frac{\mathcal{L}}{c} I_s$, where $\mathcal{L} \sim \pi R$ is the self-inductance of the system. Flux magnitude can be estimated as

$$\Phi_s/\Phi_0 \approx \frac{1}{4\sqrt{2}}\frac{N_\perp}{\zeta^2}\frac{R}{W}. \quad (17)$$

The phase difference $\phi = \chi_1 - \chi_2$ is canonically conjugate to the operator of difference of Cooper pairs' numbers in the superconducting banks, $\Delta\hat{n} = \hat{n}_1 - \hat{n}_2$ [9]. Therefore in the presence of charging energy due to electron transfer between the banks in an isolated junction (with classical capacitance C), ϕ ceases to be a constant of motion. Its evolution can be mapped on the motion of a quantum particle in a one-dimensional ring $-\pi \leq \phi < \pi$ (see e.g. [9] and references therein), with the Hamiltonian

$$\mathcal{H}(\phi, \frac{\partial}{\partial\phi}) = \frac{\hbar^2}{2M_Q}(\frac{1}{i}\frac{\partial}{\partial\phi})^2 + U(\phi); \quad M_Q = \frac{C\hbar^2}{16e^2} \equiv \frac{\hbar^2}{8\varepsilon_Q}. \quad (18)$$

The first term describes the charging energy. The effective potential $U(\phi)$ (Fig.2) corresponds to the Josephson energy:

$$U(\phi) = \frac{\hbar I_0}{4\pi^2 e}(|\phi| - \phi_0)^2 = \frac{N_\perp\bar{\epsilon}}{4\pi^2}(|\phi| - \phi_0)^2 = \frac{M_Q\omega_0^2}{2}(|\phi| - \phi_0)^2, \quad (19)$$

where $\bar{\epsilon}$ was introduced earlier, and $\omega_0 = \sqrt{\left(\frac{C\hbar^2}{16e^2}\right)^{-1} \cdot \frac{\hbar I_0}{2\pi^2 e}} = \sqrt{\frac{8}{\pi^2}N_\perp \cdot \frac{\varepsilon_Q\bar{\epsilon}}{\hbar^2}}$ is the frequency of small oscillations in a separate well. The dynamics of the system is thus determined by four parameters: ε_Q, $\bar{\epsilon}$, $N_\perp$, and ϕ_0. We will consider the case $|\phi_0| \ll \pi$ (or $|\pi - \phi_0| \ll \pi$, in which case we should take $0 \leq \phi < 2\pi$). Then in the limit $\hbar\omega_0, k_B T \ll U(0)$, we can use the two-well approximation [10], and the transition rate between the wells will be given by $\Gamma = \nu_A + \nu_T$, where the rate for thermally activated transitions $\nu_A \sim \omega_0 e^{-\beta U(0)}$, and for quantum tunneling $\nu_T \sim \omega_0 e^{-\frac{U(0)}{\hbar\omega_0}}$.

The phase fluctuations create finite voltage, $V = \frac{\hbar}{2e}\dot{\phi}$,, and therefore normal current GV flows in the normal region (its conductance $G \sim N_\perp\frac{e^2}{\pi\hbar}$). The corresponding dissipative function and the decay decrement are

$$\mathcal{F} = \frac{1}{2}\frac{d}{dt}E = \frac{1}{2}GV^2 = \frac{1}{2}G\left(\frac{\hbar}{2e}\right)^2\dot{\phi}^2; \quad \gamma = \frac{1}{M_Q\dot{\phi}}\frac{\partial\mathcal{F}}{\partial\dot{\phi}} = \frac{G\hbar^2}{4e^2 M_Q} = \frac{4}{\pi}N_\perp\frac{\varepsilon_Q}{\hbar}. \quad (20)$$

The character of quantum fluctuations of the magnetic flux drastically depends on dissipation. In the limit $\hbar\omega_0 \ll U(0)$ the conclusions basically reduce to the following [13]. The tunneling rate exponent multiplies by $\sim (1 + \gamma/\omega_0)$. Therefore if $\gamma > \omega_0$, tunneling is suppressed. (Strictly speaking, in this case the description in terms of initial two-well potential becomes meaningless.) If, on the other hand,

$$\frac{\gamma}{\omega_0} = \sqrt{N_\perp\frac{\varepsilon_Q}{\bar{\epsilon}}} \ll 1, \quad (21)$$

quantum tunneling is possible, but quantum coherence between the states in left and right wells is destroyed [11].

The problem is simplified due to phase dependence of the spontaneous current (16), which can be approximated by

$$\Phi(\phi) = \Phi_s \operatorname{sgn}\phi (\operatorname{mod} 2\pi). \quad (22)$$

This allows to describe the dynamics of Φ in terms of a two-level system [12]: state "$\downarrow$" is when ϕ is in the left well ($\Phi = -\Phi_s$), state "$\uparrow$" when ϕ is in the right well ($\Phi = \Phi_s$). Therefore we can write a set of coupled master equations for diagonal terms of density matrix ([14], Ch.7):

$$
\begin{cases} \frac{\partial}{\partial t}\rho_\uparrow = \Gamma(\rho_\downarrow - \rho_\uparrow) \\ \frac{\partial}{\partial t}\rho_\downarrow = -\Gamma(\rho_\downarrow - \rho_\uparrow) \end{cases} \tag{23}
$$

Here $\rho_{\uparrow,\downarrow}$ gives the probability of occupation of right (left) well (spontaneous flux $\pm\Phi_s$). Eq.(23) describes random telegraph noise of spontaneous magnetic flux, with autocorrelation function and spectral density [15]

$$
\langle \Phi(t)\Phi(0)\rangle = \Phi_s^2 e^{-2\Gamma|t|}; \ \left(\Phi^2\right)_\omega = \frac{4\Phi_s^2\Gamma}{\omega^2+4\Gamma^2}. \tag{24}
$$

The noise is temperature dependent at $T > T^* = \hbar\omega_0/k_B$, where thermally activated transitions are more prominent than tunneling. Below T^*, the activated processe freeze out, and quantum tunneling remains the sole source of flux noise. Magnetic noise intensity detected at a certain frequency Ω, as Γ is changed with temperature, will peak at $\Gamma(T) = \Omega/2$.

Let us make some numerical estimates. Consider the case when the role of the normal part of the system is played by 2D electron gas of a GaAs-AlGaAs heterostructure, with $\lambda_F = 450$Å and $v_F = 2\times 10^7$ cm/sec. Then the number of transverse modes is $N_\perp = 4\pi R/\lambda_F$, and the condition of "small" contact (2) becomes $N_\perp < 2\zeta\sqrt{W/\Lambda} \approx 250\sqrt{W/\Lambda}$. It will be satisfied if e.g. $N_\perp = 200, \ (R \approx 7000$ Å $), \ W = 1000$ Å. The interlevel spacing is then $\bar\epsilon \approx 10^{-15}$erg. The corresponding spontaneous flux amplitude $\Phi_s \approx 0.02\Phi_0$ is within the experimental capabilities.

The applicability condition of the two-well model imposes the condition $\varepsilon_Q/\bar\epsilon \ll (N_\perp\phi_0^4)/128\pi^2$, which will be satisfied if $\phi_0 > (128\pi^2/N_\perp)^{1/4} N_\perp^{-1/4} \approx 0.42$.

The weak dissipation condition (21) translates into $\epsilon_Q \gg 5\times 10^{-18}$erg (that is, total capacitance of the system should exceed $\sim 3\times 10^{-14}$F). This yields for $\hbar\omega_0 \ll \sqrt{\frac{8}{\pi^2}\bar\epsilon^2} \approx \bar\epsilon$, that is, $\omega_0 \ll 10^{12}$ sec^{-1}. The flux noise characteristic frequency Γ is exponentially dependent on ϕ_0^2 (through $U(0)$), and scales with temperature from $\nu_T \sim \omega_0 \exp[-U(0)/\hbar\omega_0]$ at $T \ll T^*$ to $\nu_T \sim \omega_0$ at $T \gg T^*$, and will probably reach a fraction of GHz.

In conclusion, we have found the Josephson current and spontaneous magnetic flux in an annular SND junction. The magnitude of the flux is not quantized; it depends on the parameters of the system and is of the order of few percent of Φ_0 for a typical configuration.

We demonstrated that in an isolated system, due to charging effects, there will be quantum noise of the spontaneous magnetic flux, with characteristic frequency strongly dependent on the configuration of the system and its surroundings. It can reach a fraction of GHz. The noise intensity at a fixed frequency will be a non-monotone function of temperature, allowing to measure the transition rate $\Gamma(T)$ directly and thus separate the contributions from thermally activated processes and quantum tunneling.

We hope that such systems provide experimental observation of a novel dynamical effect in *d*-wave superconductivity.

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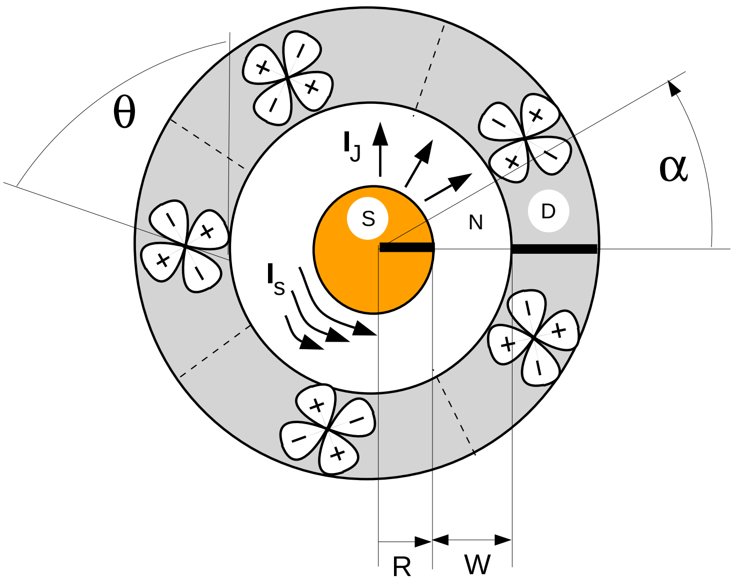

FIG. 1. Annular SND junction. The c-axis of the d-wave superconductor is chosen to be parallel to the SD boundary; θ is the angle between the SD boundary and the nodal plane of the d-wave order parameter ($0 \leq \theta \leq \frac{\pi}{2}$).

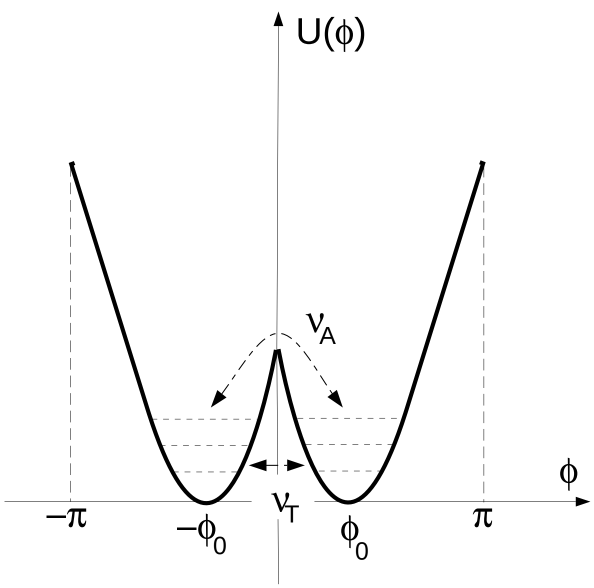

FIG. 2. Effective potential and transition rates for the phase variable in an annular SND junction.